Multi-directional higher-order amplitude squeezing

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Abstract

Fan-even K-quantum nonlinear coherent states are introduced and higher-order amplitude squeezing is investigated in such states. It is shown that for a given K the lowest order in which an amplitude component can be squeezed is 2K and the squeezing appears simultaneously in K directions separated successively in phase by π/K.

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1. Introduction

During a few last years, there has been growing interest in the nonlinear coherent state (NCS) [1–5] defined as the eigenstate |χ; f⟩ of the non-boson non-Hermitian operator af(â),

af(â)|χ; f⟩ = χ|χ; f⟩,

(1)

with â = a†a, a the bosonic annihilation operator, χ a complex eigenvalue and f an arbitrary (assumed to be real) nonlinear operator-valued function of â. Recently, the so-called K-quantum nonlinear coherent state (KNCS) has been introduced [6–8] as a generalization of the NCS to K (K an arbitrary positive integer) eigenstates |ξ; K, j, f⟩ of the non-boson non-Hermitian operator aKf(â),
aKf(â)|ξ; K, j, f⟩ = ξK|ξ; K, j, f⟩,

(2)

with j = 0, 1, . . . , K − 1 and ξ a complex number. Note that the notations in [6–8] are not the same. In [6] (af(â))k is used instead of aKf(â). In [7] it is aK+1f(â) with N counted from zero and the state is named NCS of order N + 1. The even (odd) NCS [9–11] corresponds to K = 2, j = 0 (j = 1). The linear case (see, e.g., [12–18]) is recovered when f ≡ 1. Other types of multi-quantum states are also developed (see, e.g., [19,20]). In [7,8] the KNCS has been shown physically realizable in the quantized vibration of the center-of-mass motion of a harmonically trapped ion which is further properly driven by two laser beams one of which is resonant and the other is detuned to the Kth lower sideband. The KNCS displays a multi-peaked structure in the number distribution depending on both the character of the nonlinear function f and the power K. The normalization difficulty connected with the wildly oscillating behavior at large n and Lamb–Dicke parameter has been dealt with in [7] while nonclassical effects have been studied in [8]. In this Letter we further explore the KNCS under another angle of view. Namely, a property of the KNCS will be used to construct the so-called fan-even K-quantum nonlinear coherent states (FEKNCS’s) in

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which higher-order amplitude squeezing will be investigated. Unlike in usual states, in FEKNCS’s multi-
directional squeezing is possible.

2. Fan-even K-quantum nonlinear coherent state

Spanned in Fock space the KNCS has the explicit form

\[ |\xi; K, j, f\rangle = C_{Kj} \sum_{n=0}^{\infty} \frac{\xi^{nK+j}}{\sqrt{(nK+j)!} f(nK+j)!} |nK+j\rangle, \tag{3} \]

where

\[ f(nK+j) = \prod_{q=0}^{n} f(qK+j), \quad n \geq 1, \quad n = 0, \tag{4} \]

and the coefficient \( C_{Kj} \) is determined as

\[ C_{Kj} = C_{Kj}(|\xi|^2) = \left[ \sum_{m=0}^{\infty} \frac{|\xi|^2(mK+j)}{(mK+j)! f(mK+j)!} \right]^{-1/2} \tag{5} \]

in order to normalize the KNCS to 1. Among the various properties of the KNCS (see [6–8]), the one of our direct concern here is that any state \( |\xi; K, j, f\rangle \) can be decomposed into a correlated combination of \( K \) states \( |\chi_l; f\rangle \) as

\[ |\xi; K, j, f\rangle = \frac{1}{K C_{10}(|\xi|^2)} C_{Kj}(|\xi|^2) \sum_{l=0}^{K-1} \exp\left(-\frac{2\pi i l}{K} j\right) |\chi_l; f\rangle \tag{6} \]

with \( |\chi_l; f\rangle \) the NCS defined by Eq. (1) and

\[ \chi_l = \xi \exp\left(\frac{2\pi i l}{K}\right). \tag{7} \]

Decomposition (6) can be verified straightforwardly by substituting \( |\chi_l; f\rangle \equiv |\chi_l; 1, 0, f\rangle \) and (7) into the r.h.s. of (6) with subsequent use of the identity

\[ \sum_{l=0}^{L-1} \exp\left(\frac{2\pi i q l}{L}\right) = L \delta_{q,mL}. \tag{8} \]

with \( m \) a nonnegative integer. Eq. (6) represents a superposition state belonging to the type of generalized NCS,

\[ |\Psi_K\rangle = \sum_{l=0}^{K-1} a_l^{(K)} |\xi\exp(i\omega_l^{(K)})\rangle f\rangle, \tag{9} \]

whose linear version with \( f \equiv 1 \) was found in [21] and further developed in [22,23] (see also [24,25] and references therein for general types of superposition states, still with \( f \equiv 1 \)). Generally, by choosing suitably the weights \( a_l^{(K)} \) and phases \( \omega_l^{(K)} \) the state \( |\Psi_K\rangle \) can be tailored for various kinds of state. The choice \( a_l^{(K)} = 2\pi l/K \) and \( \omega_l^{(K)} = C_{Kj} \exp(-2\pi i jl/K)/ (KC_{10}) \) fits \( |\Psi_K\rangle \) to be the normalized KNCS \( |\xi; K, j, f\rangle \). The ends of \( \chi_l \). Eq. (7), are equi-distantly spaced along a circle of radius \( |\xi| \) in the complex plane and \( |\xi; K, j, f\rangle \) can be referred to as state on a circle [26] or circular state [27].

Let \( T_m \) be the rotation operator that rotates the \( \chi_l \) on an angle \( \phi_m = 2\pi m/K \) \( (m = 0, 1, \ldots, K-1) \), i.e.,

\[ T_m|\chi_l; f\rangle = |\chi_l'; f\rangle = |\exp\left(\frac{2\pi i m}{K}\right)\chi_l; f\rangle. \tag{10} \]

Under such a rotation the KNCS gains a \( j \)-dependent phase shift

\[ T_m|\xi; K, j, f\rangle = |\chi; \frac{2\pi i m}{K}\rangle \tag{11} \]

as is evident from (9). Transformation (10) indicates that in general the KNCS \( |\xi; K, j, f\rangle \) cannot be identified as even or odd in the usual sense with respect to the inversion \( \xi \rightarrow -\xi \) corresponding to a \( \pi \)-rotation. More precisely, for odd \( K \) the KNCS is neither even nor odd. However, for even \( K \) the states \( |\xi; K, j, f\rangle \) may be either even or odd depending on the evenness of \( j \). If \( j \) is even (odd) the KNCS is even (odd) too. Moreover, when \( j = 0 \) the states \( |\xi; K, 0, f\rangle \) for any \( K \) turn out to be symmetric in the sense of their invariance under the rotation \( T_m \) with any \( m = 0, 1, \ldots, K-1 \), as is transparent from (10).

If, in addition to \( j = 0 \), \( K \) is even then the states become at the same time both symmetric and even. In what follows we shall need such symmetric-even \( K \)-quantum nonlinear coherent states (SEKNCS’s), i.e., the states \( |\xi; K, 0, f\rangle \) with even \( K \), and denote them.
by \(|\xi; K, f\rangle_{se}\),

\[|\xi; K, f\rangle_{se} = \frac{1}{K} \sum_{i=0}^{K-1} C_{K0}(|\xi|^2) \sum_{l=0}^{K} (\chi_l; f),\]  

(11)

with \(K\) even, where the subscript “se” stands for simultaneous “symmetric” and “even”. “Simultaneous” is crucial because the KNCS may be symmetric but not even (e.g., for \(j = 0, K\) odd) or even but asymmetric (e.g., \(K\) even, \(j = 2, 4, \ldots, K - 2 \neq 0\)). In particular, when \(K = 2\) the states \(|\xi, 2\rangle_{se}\) are simply referred to as even nonlinear coherent states since in this special case “symmetric” and “even” coincide. Fig. 1(a) represents the orientation of the \(\chi_l\) in the complex plane in a SEKNCS with \(K = 8\). For an odd \(K\) the \(\chi_l\) point symmetrically too but the state made of them is neither even nor odd (see Fig. 1(b), for \(K = 7\)).

We now construct a state denoted by \(|\xi; K, f\rangle_F\) which is superposed by SEKNCs’s \(|\xi_q; K, f\rangle_{se}\) in the following way:

\[|\xi; K, f\rangle_F = B_K \sum_{q=0}^{K-1} |\xi_q; K, f\rangle_{se}.\]  

(12)

In definition (12),

\[\xi_q = \xi \exp\left(\frac{\pi i q}{K}\right),\]  

(13)

and \(B_K \equiv B_K (|\xi|^2)\) is the normalization coefficient determined from the equation

\[B_K^2 (|\xi|^2) C_{K0}^2 (|\xi|^2) D_K (|\xi|^2) = 1,\]  

(14)

where

\[D_K (|\xi|^2) = \sum_{m=0}^{\infty} \frac{|\xi|^{2mK} |JK(m)|^2}{(mK)! [f(mK)]^2}\]  

(15)

with \(JK(m)\) given by

\[JK(m) = \sum_{q=0}^{K-1} \exp(i\pi q m).\]  

(16)

The state \(|\xi; K, f\rangle_F\) defined by Eq. (12) is in fact a \(K\)-quantum superposition state made of the component states \(|\xi_q; K, f\rangle_{se}\) which is also a \(K\)-quantum superposition state over the simpler single-quantum states \(|\chi_l; f\rangle \equiv |\chi_l; 1, f\rangle\). In this respect \(|\xi_q; K, f\rangle_{se}\) can be considered as the primary superposition states and \(|\xi; K, f\rangle_F\) the secondary ones. This makes sense toward a production scheme: the primary superposition states should be generated first from the single-quantum states and then, in the second step, these are to be used as the input for the secondary superposition state as the output. The orientation of \(\xi_q\), Eq. (13), in the complex plane (Fig. 2) looks like an open paper fan. Hence we call the state \(|\xi; K, f\rangle_F\) fan-even \(K\)-quantum nonlinear coherent states (FEKNCs’s) with the subscript “\(F\)” standing for “fan”. In what follows FEKNCs’s will be referred in short to as fan-states. When \(K = 2\) the fan shrinks to a setsquare (une equerre). That is why the state

\[|\xi; 2, f\rangle_F = B_2 \left(|\xi; 2, f\rangle_{se} + |i\xi; 2, f\rangle_{se}\right)\]  

(17)

was named orthogonal-even nonlinear coherent state [28] which is the simplest fan-state. In addition, if \(f = 1\), state (17) reduces to that proposed in [29].

3. Multi-directional higher-order amplitude squeezing

Consider a boson field with the annihilation and creation operators \(a\) and \(a^\dagger\) obeying the Bose–Einstein
commutation relation \([a, a^+] = 1\). Let \(X_\varphi\) be a field amplitude component pointing along the direction making an angle \(\varphi\) with the real axis in the complex plane
\[
X_\varphi = \frac{(ae^{-i\varphi} + a^+e^{i\varphi})}{\sqrt{2}}.
\]
The \(\sqrt{2}\) was used above instead of 2 in the definition of \(X_\varphi\) is just a matter of notation. A state \(|\ldots\rangle\) is said to be amplitude-squeezed to the \(N\)th order \((N = 2, 4, 6, \ldots)\) along the direction \(\varphi\) if the quantity \(S_{\varphi,N}\),
\[
S_{\varphi,N} = [(\Delta X_\varphi)^N] - (N - 1)!/\sqrt{2^N},
\]
with \(\Delta X_\varphi \equiv X_\varphi - \langle X_\varphi \rangle\), gets negative [30].

Let the real axis be chosen along the direction of \(\xi\).

This allows treating \(\xi\) as a real number. We now proceed to study in detail various higher-order amplitude squeezings in the above constructed fan-states \(|\xi; K, f\rangle\). The problem depends essentially on the concrete form of the nonlinear function \(f(\hat{n})\) which differs strongly from one to another physical context (e.g., see [31] for harmonious oscillators, [32] for photon-added coherent states, [1,8,33] for trapped ions, etc.) and requires formidable numerical simulations. Nevertheless, it is worth stressing that our main target here is to show a possible multi-directional character of squeezing in the fan-state. Since \(f\) depends only on \(\hat{n}\), i.e., it is phase-independent, the squeezing directions are not affected by \(f\). Its inclusion due to its nonlinearity will only modify the parameter space where squeezing may appear. By this reason and to pursue our primary goal we limit ourselves in this Letter to \(f \equiv 1\) which allows us to achieve the general results at a fully analytic level. The power \(K\) is, however, kept arbitrary.

For \(K = N = 2\) we have obtained
\[
S_{\varphi,2}^{(K=2)} = \frac{\xi^2}{D_2} \left[ \sinh (\xi^2) - \sin (\xi^2) \right]
\]
with
\[
D_2 = \cosh(\xi^2) + \cos(\xi^2).
\]

Since \(S_{\varphi,2}^{(K=2)}\) is independent of \(\varphi\) and always positive, the conventional amplitude squeezing \((N = 2)\) is totally absent.

For \(K = 2\) and \(N = 4\) we have obtained
\[
S_{\varphi,4}^{(K=2)} = \frac{\xi^2}{2} \left[ \frac{\xi^2}{2} \cos(4\varphi) - \frac{3}{D_2} \left[ \xi^2 (\cos(\xi^2) - \cosh(\xi^2)) \right. \\
+ 2 \left. (\sin(\xi^2) - \sinh(\xi^2)) \right] \right].
\]

It follows from (22) that the fourth-order squeezing occurs whenever
\[
\cos(4\varphi) < g(\xi) = \frac{3 \left( \xi^2 \cos(\xi^2) - \cosh(\xi^2) + 2(\sin(\xi^2) - \sinh(\xi^2)) \right)}{\xi^2 (\cosh(\xi^2) - \cosh(\xi^2))}.
\]

The \(g(\xi)\) equals zero at \(\xi = 0\). As \(\xi\) increases, \(g\) first decreases then, after reaching a minimum, increases up and tends to \(3\) when \(|\xi| \to \infty\). No squeezing appears for \(\xi > \xi_c = 0.796541\) for which \(g(\xi) \leq -1\) and no \(\varphi\) can be found to make \(S_{\varphi,4}^{(K=2)}\) negative. Yet, for \(0 < |\xi| < \xi_c\) there exist intervals of \(\varphi\) that fulfills condition (23). Let us define the squeezing (stretching) direction as that along which the amplitude is maximally squeezed (unsqueezed). Then it can be verified that the squeezing directions are along \(\varphi = \varphi_{sq,1} = \pi/4\) \((5\pi/4)\) and \(\varphi = \varphi_{sq,2} = 3\pi/4\) \((7\pi/4)\) while for the stretching directions \(\varphi = \varphi_{st,1} = 0\) \((\pi)\) and \(\varphi = \varphi_{st,2} = \pi/2\) \((3\pi/2)\). Along a squeezing direction a maximal squeezing is reached at \(|\xi| = \xi_M = 0.669272\). The existence of two squeezing directions that are orthogonal to each other is owing to the symmetry of the fan-state under consideration with \(K = 2\).
For $K = 2$ and $N = 6$ we have obtained
\[
S^{(K=2)}_{0,6} = \frac{\xi^2}{4} \left\{ \left[ \frac{15}{2} + \frac{3\xi^2}{2D_2} (\sinh(\xi^2) - \sin(\xi^2)) \right] \times \cos(4\varphi) \\
+ \frac{1}{2D_2} \left[ 10\xi^4 (\sinh(\xi^2) + \sin(\xi^2)) \\
+ 45\xi^2 (\cosh(\xi^2) - \cos(\xi^2)) \\
+ 45 (\sin(\xi^2) - \sin(\xi^2)) \right] \right\}.
\]

(24)

Fig. 3(a) is a 3D plot of $S^{(K=2)}_{0,6}$ as a function of $|\xi|$ and $\varphi$ showing again maximal squeezing (stretching) occurred along the directions $\varphi = \varphi_{sq,1}$ and $\varphi = \varphi_{sq,2}$ ($\varphi = \varphi_{sq}$). The values of $|\xi|$ for which squeezing appears are confined within the interval $|\xi| = 0.785486$. The two coexistent directions of squeezing are most visualized by a polar plot (Fig. 3(b)) where the four-winged flower is deformed into a four-winged flower. More flower-like is the $S^{(K=2)}_{0,6}$ itself when it is watched upon in polar coordinates (Fig. 3(c)).

For $K = 4$ and $N = 2, 4, 6$ we have obtained
\[
S^{(K=4)}_{0,2} = \frac{\xi^2}{4D_4} \left\{ \sinh(\xi^2) - \sin(\xi^2) \\
+ \sqrt{2} \left[ \sinh \left( \frac{\xi^2}{\sqrt{2}} \right) \cos \left( \frac{\xi^2}{\sqrt{2}} \right) \\
- \sin \left( \frac{\xi^2}{\sqrt{2}} \right) \cos \left( \frac{\xi^2}{\sqrt{2}} \right) \right] \right\},
\]

(25)
\[
S^{(K=4)}_{0,4} = \frac{3\xi^2}{4D_4} \left\{ \xi^2 \left[ \cosh(\xi^2) - \cos(\xi^2) \\
- 2 \sinh \left( \frac{\xi^2}{\sqrt{2}} \right) \sin \left( \frac{\xi^2}{\sqrt{2}} \right) \right] \\
+ \sinh(\xi^2) - \sin(\xi^2) \\
+ \sqrt{2} \left[ \sinh \left( \frac{\xi^2}{\sqrt{2}} \right) \cos \left( \frac{\xi^2}{\sqrt{2}} \right) \\
- \sin \left( \frac{\xi^2}{\sqrt{2}} \right) \cos \left( \frac{\xi^2}{\sqrt{2}} \right) \right] \right\},
\]

(26)
\[
S^{(K=4)}_{0,6} = \frac{5\xi^2}{4D_4} \left\{ 2\xi^4 \left[ \sinh(\xi^2) + \sin(\xi^2) \\
- \sqrt{2} \right] \left[ \sinh \left( \frac{\xi^2}{\sqrt{2}} \right) \cos \left( \frac{\xi^2}{\sqrt{2}} \right) \\
+ \sin \left( \frac{\xi^2}{\sqrt{2}} \right) \cos \left( \frac{\xi^2}{\sqrt{2}} \right) \right] \\
+ 9\xi^2 \left[ \cosh(\xi^2) - \cos(\xi^2) \\
- 2 \sinh \left( \frac{\xi^2}{\sqrt{2}} \right) \sin \left( \frac{\xi^2}{\sqrt{2}} \right) \right] \\
+ 9 \left[ \sinh(\xi^2) - \sin(\xi^2) \\
+ \sqrt{2} \left[ \sinh \left( \frac{\xi^2}{\sqrt{2}} \right) \cos \left( \frac{\xi^2}{\sqrt{2}} \right) \\
- \sin \left( \frac{\xi^2}{\sqrt{2}} \right) \cos \left( \frac{\xi^2}{\sqrt{2}} \right) \right] \right\}.
\]

(27)

The $S^{(K=4)}_{0,6}$ are all independent of $\varphi$ and always positive resulting in no squeezing.

For $K = 4$ and $N = 8$ we have obtained
\[
S^{(K=4)}_{0,8} = \frac{\xi^2}{8} \left\{ \xi^6 \cos(8\varphi) \\
+ \frac{1}{D_4} \left[ 35\xi^6 \left( \cosh(\xi^2) + \cos(\xi^2) \\
- 2 \cosh \left( \frac{\xi^2}{\sqrt{2}} \right) \cos \left( \frac{\xi^2}{\sqrt{2}} \right) \right] \\
+ 280\xi^4 \left[ \sinh(\xi^2) + \sin(\xi^2) \\
- \sqrt{2} \sinh \left( \frac{\xi^2}{\sqrt{2}} \right) \cos \left( \frac{\xi^2}{\sqrt{2}} \right) \\
- \sqrt{2} \cosh \left( \frac{\xi^2}{\sqrt{2}} \right) \sin \left( \frac{\xi^2}{\sqrt{2}} \right) \right] \\
+ 622\xi^2 \left( \cosh(\xi^2) - \cos(\xi^2) \\
- 2 \sinh \left( \frac{\xi^2}{\sqrt{2}} \right) \sin \left( \frac{\xi^2}{\sqrt{2}} \right) \right) \right\}.
\]

(28)
Fig. 3. (a) $S = S_{\psi, 6}^{(K=2)}$ as a function of $|\xi|$ and $\varphi$. (b) The uncertainty domain in the coherent state (dashed circle) and in the fan-state (solid four-winged flower) with $K = 2$ for $|\xi| = 0.659657$. It is visual that, though squeezed in two directions, the area bounded by the flower is larger than that bounded by the circle. (c) $S_{\psi, 6}^{(K=2)}$ in polar coordinates for the same value of $|\xi|$ as in (b). The small wings correspond to squeezing, the big ones to stretching and the center to the coherent state. The distance from the center to the farthest point of a big wing is 1.07.
the two quadratures

\[ S(K=4) \text{ as a function of } \varphi \text{ for three selected values of } |\xi|. \]

From the figure, it is clear that maximal squeezing occurs along the four directions

\[ \varphi = \varphi_{\text{sq},1} = \pi/8 \ (9\pi/8), \varphi = \varphi_{\text{sq},2} = 3\pi/8 \ (11\pi/8), \]

\[ \varphi = \varphi_{\text{sq},3} = 5\pi/8 \ (13\pi/8) \text{ and } \varphi = \varphi_{\text{sq},4} = 7\pi/8 \ (15\pi/8), \]

whereas for maximal stretching \( \varphi = \varphi_{\text{st},1} = 0 \ (\pi), \varphi = \varphi_{\text{st},2} = \pi/4 \ (5\pi/4), \varphi = \varphi_{\text{st},3} = \pi/2 \ (3\pi/2) \text{ and } \varphi = \varphi_{\text{st},4} = 3\pi/4 \ (7\pi/4). \]

The values of \( |\xi| \) for which squeezing appears lie within the interval [0, 0.823267] and a maximal squeezing is reached at \( |\xi| = 0.754939 \).

Expressions of amplitude squeezing for higher values of \( K \) and \( N \) have also been obtained which are much more cumbersome. The general results inferred from their detailed analysis will be drawn below. Before doing so let us address a delicate issue. By construction, fan-states display the same property along directions separated by a multiple of \( \pi/K \). As \( K \) is even, two quadratures are always among the equivalent directions. Therefore, if squeezing appears along a direction \( \varphi_{\text{sq},1} \), then the one that is perpendicular to it, \( \varphi_{\text{sq}+\pi/2} \), is also equally squeezed. The product of the two quadratures \( \langle (\Delta X_{\varphi_{\text{sq}}})^N \rangle_{\text{CS}} \langle (\Delta X_{\varphi_{\text{sq}+\pi/2}})^N \rangle_{\text{CS}} \) is obviously less than

\[
\langle (\Delta X_{\varphi_{\text{sq}}})^N \rangle_{\text{CS}} \langle (\Delta X_{\varphi_{\text{sq}+\pi/2}})^N \rangle_{\text{CS}} = (N - 1)!/\sqrt{2^N} R_{\text{CS}}^2\]

Does this mean that the coherent state does not minimize the uncertainty product as was questioned in [34]? It was partly answered in [35] where the authors opened the idea that in such situations a pair of quadratures are no longer the right canonical conjugates meant in the Heisenberg inequality and suggested choosing two field components that differ in phase by less than \( \pi/2 \) as more appropriate as canonical conjugates. In our fan-states the two amplitude components, one along a squeezing direction, the other along the next stretching direction, are best candidates for the right conjugates. Yet, whether or not is the CS a min-
In our opinion, the uncertainty is best evaluated by its area rather than product. The uncertainty area, \( A_{N}^{(K)} \), is given by
\[
A_{N}^{(K)} = \frac{1}{2} \int_{0}^{2\pi} d\phi ((\Delta X_{\phi})^{N})^{2}\frac{m}{F}
\]
\[
= \pi \left( R_{N}^{2} + (2R_{N} + X_{N}^{(K)})X_{N}^{(K)} + \frac{1}{2} \sum_{p=1}^{[N/2]} (Y_{N}^{(K)}(p))^{2} \right),
\]
where \([x]\) equals the integer part of \( x \),
\[
X_{N}^{(K)} = \frac{N}{2\pi} \sum_{m=1}^{N/2} \frac{2m(a^{m}d^{m})F}{(m)^{2}(N/2 - m)!}
\]
which is always positive and
\[
Y_{N}^{(K)}(p) = \frac{2^{pK}N!}{2N} \times \sum_{m=0}^{[N/2-pK]} \frac{2m(a^{m}d^{m+2pK})F}{m!(m+2pK)(N/2 - m - pK)!}
\]
Because \( \pi R_{N}^{2} \) is the circle area of uncertainty in the CS, \( X_{N}^{(K)}(p) \) is squared and \( X_{N}^{(K)} \) ≥ 0 the uncertainty in the fan-states is always greater than that in the coherent state. Hence, the coherent state remains a right MUS as far as the fan-states are concerned. Problems arising from more subtle questions such as how the fan-states show up with the Schrödinger–Robertson uncertainty relation, whether the coherent state is an intelligent state for higher-order moments, etc. are certainly beyond this Letter scope. On this topic, recommended is a recent paper [36] which contains detailed discussions as well as a good list of references.

4. Conclusion

In conclusion, the fan-state has been constructed as a superposition state of the symmetric-even \( K \)-quantum nonlinear coherent states which in turn are superposition states of the more elementary single-quantum nonlinear coherent states. Such a secondary superposition provides the fan-state with the symmetry leading to multi-directional squeezing. In these fan-states higher-order amplitude squeezing has been calculated analytically for a wide set of \( K \) and \( N \). The general results are summarized as follows. For a fixed \( K \) an amplitude component cannot be squeezed at all in orders \( N \) less than 2\( K \). The lowest order in which squeezing may appear is \( N_{\text{min}} = 2K \). Squeezing may also occur in an arbitrary (even) order greater than \( N_{\text{min}} \). That is, given \( K \) the orders in which amplitude squeezing can be observed is \( N_{\text{sque}} = 2(K + n) \) with \( n = 0, 1, 2, \ldots \). For fixed \( K \) and \( N_{\text{sque}} \), squeezing, whenever it exists, is \( K \)-directional, i.e., it occurs equally along \( K \) directions determined by the angles
\[
\psi = \psi_{\text{sque},m} = [(1 + 2m)\pi]/(2K)
\]
with \( m = 0, 1, \ldots, K - 1 \), while for stretching directions \( \psi = \psi_{\text{sque},m} = \pi m/K \). The uncertainty domain in the complex plane has the shape of a 2\( K \)-winged flower (see, e.g., Fig. 3(b)) and the squeeze parameter \( S_{\psi,N_{\text{sque}}} \) itself, in polar coordinates, shows up as a 4\( K \)-winged flower (see, e.g., Figs. 3(c) and 4(b)) with 2\( K \) small wings associated with squeezing and 2\( K \) big wings associated with stretching. In the fan-state, which prompts higher-order squeezing, the two quadratures can no longer be served as the right pair of canonical conjugates. The appropriate ones should be two amplitude components which are \( \pi/(2K) \) dephased from each other (i.e., one component points along a squeezing direction and the other component points along the stretching direction next to the squeezing one). Though being squeezed in more than one direction the uncertainty area associated with the fan-state is found always larger than that in the coherent state revealing the relevance of the coherent state as a MUS.

Multi-quantum states associated with higher-order squeezing have been studied intensively in the literature. For example, linear four-photon states were investigated in [15] and squeezing was calculated up to order \( N = 8 \). Yet, squeezing was found to occur only in one direction. In this Letter we concentrate on the multi-direction possibility of squeezing rather than on the nonlinearity brought in by the function \( f \). As was mentioned above, the number of squeezing directions in fan-states does not depend on whether \( f = 1 \) or \( f \neq 1 \). The formalism has, however, been formulated
explicitly in the evident presence of $f$ and ready for its actual inclusion. The shape of uncertainty flower would be dramatically modified (the number of flower wings is preserved for a fixed $K$) and even the possible existence of the fan-states themselves should be analyzed carefully with each specific type of $f \neq 1$ in dependence on the parameters involved. This would constitute an intriguing piece of research and will be done in the future.

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