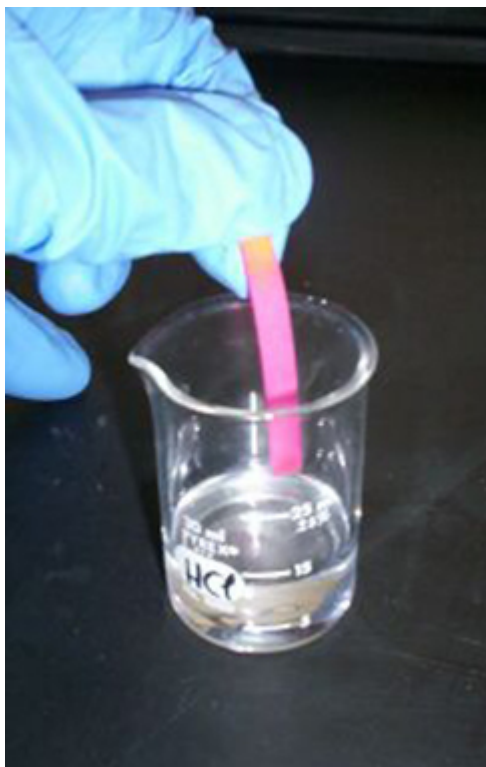




# Logarithmic Properties

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*The pH of hydrochloric acid is tested with litmus paper. (credit: David Berardan)*

In chemistry, pH is used as a measure of the acidity or alkalinity of a substance. The pH scale runs from 0 to 14. Substances with a pH less than 7 are considered acidic, and substances with a pH greater than 7 are said to be alkaline. Our bodies, for instance, must maintain a pH close to 7.35 in order for enzymes to work properly. To get a feel for what is acidic and what is alkaline, consider the following pH levels of some common substances:

- Battery acid: 0.8
- Stomach acid: 2.7
- Orange juice: 3.3
- Pure water: 7 (at 25° C)
- Human blood: 7.35
- Fresh coconut: 7.8

- Sodium hydroxide (lye): 14

To determine whether a solution is acidic or alkaline, we find its pH, which is a measure of the number of active positive hydrogen ions in the solution. The pH is defined by the following formula, where  $a$  is the concentration of hydrogen ion in the solution

$$\begin{aligned}\text{pH} &= -\log([H^+]) \\ &= \log\left(\frac{1}{[H^+]}\right)\end{aligned}$$

The equivalence of  $-\log([H^+])$  and  $\log\left(\frac{1}{[H^+]}\right)$  is one of the logarithm properties we will examine in this section.

### Using the Product Rule for Logarithms

Recall that the logarithmic and exponential functions “undo” each other. This means that logarithms have similar properties to exponents. Some important properties of logarithms are given here. First, the following properties are easy to prove.

$$\log_b 1 = 0$$

$$\log_b b = 1$$

For example,  $\log_5 1 = 0$  since  $5^0 = 1$ . And  $\log_5 5 = 1$  since  $5^1 = 5$ .

Next, we have the inverse property.

$$\log_b(b^x) = x$$

$$b^{\log_b x} = x, x > 0$$

For example, to evaluate  $\log(100)$ , we can rewrite the logarithm as  $\log_{10}(10^2)$ , and then apply the inverse property  $\log_b(b^x) = x$  to get  $\log_{10}(10^2) = 2$ .

To evaluate  $e^{\ln(7)}$ , we can rewrite the logarithm as  $e^{\log_e 7}$ , and then apply the inverse property  $b^{\log_b x} = x$  to get  $e^{\log_e 7} = 7$ .

Finally, we have the one-to-one property.

## Logarithmic Properties

$$\log_b M = \log_b N \text{ if and only if } M = N$$

We can use the one-to-one property to solve the equation  $\log_3(3x) = \log_3(2x + 5)$  for  $x$ . Since the bases are the same, we can apply the one-to-one property by setting the arguments equal and solving for  $x$ :

$$3x = 2x + 5 \quad \text{Set the arguments equal.}$$

$$x = 5 \quad \text{Subtract } 2x.$$

But what about the equation  $\log_3(3x) + \log_3(2x + 5) = 2$ ? The one-to-one property does not help us in this instance. Before we can solve an equation like this, we need a method for combining terms on the left side of the equation.

Recall that we use the *product rule of exponents* to combine the product of exponents by adding:  $x^a x^b = x^{a+b}$ . We have a similar property for logarithms, called the **product rule for logarithms**, which says that the logarithm of a product is equal to a sum of logarithms. Because logs are exponents, and we multiply like bases, we can add the exponents. We will use the inverse property to derive the product rule below.

Given any real number  $x$  and positive real numbers  $M$ ,  $N$ , and  $b$ , where  $b \neq 1$ , we will show

$$\log_b(MN) = \log_b(M) + \log_b(N).$$

Let  $m = \log_b M$  and  $n = \log_b N$ . In exponential form, these equations are  $b^m = M$  and  $b^n = N$ . It follows that

$$\begin{aligned} \log_b(MN) &= \log_b(b^m b^n) && \text{Substitute for } M \text{ and } N. \\ &= \log_b(b^{m+n}) && \text{Apply the product rule for exponents.} \\ &= m + n && \text{Apply the inverse property of logs.} \\ &= \log_b(M) + \log_b(N) && \text{Substitute for } m \text{ and } n. \end{aligned}$$

Note that repeated applications of the product rule for logarithms allow us to simplify the logarithm of the product of any number of factors. For example, consider  $\log_b(wxyz)$ . Using the product rule for logarithms, we can rewrite this logarithm of a product as the sum of logarithms of its factors:

$$\log_b(wxyz) = \log_b w + \log_b x + \log_b y + \log_b z$$

A General Note

The Product Rule for Logarithms

## Logarithmic Properties

The product rule for logarithms can be used to simplify a logarithm of a product by rewriting it as a sum of individual logarithms.

$$\log_b(MN) = \log_b(M) + \log_b(N) \text{ for } b > 0$$

How To

**Given the logarithm of a product, use the product rule of logarithms to write an equivalent sum of logarithms.**

1. Factor the argument completely, expressing each whole number factor as a product of primes.
2. Write the equivalent expression by summing the logarithms of each factor.

Using the Product Rule for Logarithms

Expand  $\log_3(30x(3x + 4))$ .

We begin by factoring the argument completely, expressing 30 as a product of primes.

$$\log_3(30x(3x + 4)) = \log_3(2 \cdot 3 \cdot 5 \cdot x \cdot (3x + 4))$$

Next we write the equivalent equation by summing the logarithms of each factor.

$$\log_3(30x(3x + 4)) = \log_3(2) + \log_3(3) + \log_3(5) + \log_3(x) + \log_3(3x + 4)$$

Try It

Expand  $\log_b(8k)$ .

$$\log_b 2 + \log_b 2 + \log_b 2 + \log_b k = 3\log_b 2 + \log_b k$$

## Using the Quotient Rule for Logarithms

For quotients, we have a similar rule for logarithms. Recall that we use the *quotient rule of exponents* to combine the quotient of exponents by subtracting:  $x^{\frac{a}{b}} = x^{a-b}$ . The **quotient rule for logarithms** says that the logarithm of a quotient is equal to a difference of logarithms. Just as with the product rule, we can use the inverse property to derive the quotient rule.

Given any real number  $x$  and positive real numbers  $M$ ,  $N$ , and  $b$ , where  $b \neq 1$ , we will show

$$\log_b\left(\frac{M}{N}\right) = \log_b(M) - \log_b(N).$$

## Logarithmic Properties

Let  $m = \log_b M$  and  $n = \log_b N$ . In exponential form, these equations are  $b^m = M$  and  $b^n = N$ . It follows that

$$\begin{aligned}\log_b\left(\frac{M}{N}\right) &= \log_b\left(\frac{b^m}{b^n}\right) && \text{Substitute for } M \text{ and } N. \\ &= \log_b(b^{m-n}) && \text{Apply the quotient rule for exponents.} \\ &= m - n && \text{Apply the inverse property of logs.} \\ &= \log_b(M) - \log_b(N) && \text{Substitute for } m \text{ and } n.\end{aligned}$$

For example, to expand  $\log\left(\frac{2x^2+6x}{3x+9}\right)$ , we must first express the quotient in lowest terms. Factoring and canceling we get,

$$\begin{aligned}\log\left(\frac{2x^2+6x}{3x+9}\right) &= \log\left(\frac{2x(x+3)}{3(x+3)}\right) && \text{Factor the numerator and denominator.} \\ &= \log\left(\frac{2x}{3}\right) && \text{Cancel the common factors.}\end{aligned}$$

Next we apply the quotient rule by subtracting the logarithm of the denominator from the logarithm of the numerator. Then we apply the product rule.

$$\begin{aligned}\log\left(\frac{2x}{3}\right) &= \log(2x) - \log(3) \\ &= \log(2) + \log(x) - \log(3)\end{aligned}$$

### A General Note

#### The Quotient Rule for Logarithms

The quotient rule for logarithms can be used to simplify a logarithm or a quotient by rewriting it as the difference of individual logarithms.

$$\log_b\left(\frac{M}{N}\right) = \log_b M - \log_b N$$

#### How To

**Given the logarithm of a quotient, use the quotient rule of logarithms to write an equivalent difference of logarithms.**

## Logarithmic Properties

1. Express the argument in lowest terms by factoring the numerator and denominator and canceling common terms.
2. Write the equivalent expression by subtracting the logarithm of the denominator from the logarithm of the numerator.
3. Check to see that each term is fully expanded. If not, apply the product rule for logarithms to expand completely.

Using the Quotient Rule for Logarithms

$$\text{Expand } \log_2\left(\frac{15x(x-1)}{(3x+4)(2-x)}\right).$$

First we note that the quotient is factored and in lowest terms, so we apply the quotient rule.

$$\log_2\left(\frac{15x(x-1)}{(3x+4)(2-x)}\right) = \log_2(15x(x-1)) - \log_2((3x+4)(2-x))$$

Notice that the resulting terms are logarithms of products. To expand completely, we apply the product rule, noting that the prime factors of the factor 15 are 3 and 5.

$$\begin{aligned} \log_2(15x(x-1)) - \log_2((3x+4)(2-x)) &= [\log_2(3) + \log_2(5) + \log_2(x) + \log_2(x-1)] - [\log_2(3x+4) + \log_2(2-x)] \\ &= \log_2(3) + \log_2(5) + \log_2(x) + \log_2(x-1) - \log_2(3x+4) - \log_2(2-x) \end{aligned}$$

Analysis

There are exceptions to consider in this and later examples. First, because denominators must never be zero, this expression is not defined for  $x = -\frac{4}{3}$  and  $x = 2$ . Also, since the argument of a logarithm must be positive, we note as we observe the expanded logarithm, that  $x > 0$ ,  $x > 1$ ,  $x > -\frac{4}{3}$ , and  $x < 2$ . Combining these conditions is beyond the scope of this section, and we will not consider them here or in subsequent exercises.

Try It

$$\text{Expand } \log_3\left(\frac{7x^2+21x}{7x(x-1)(x-2)}\right).$$

$$\log_3(x+3) - \log_3(x-1) - \log_3(x-2)$$

## Using the Power Rule for Logarithms

We've explored the product rule and the quotient rule, but how can we take the logarithm of a power, such as  $x^2$ ? One method is as follows:

## Logarithmic Properties

$$\begin{aligned}\log_b(x^2) &= \log_b(x \cdot x) \\ &= \log_b x + \log_b x \\ &= 2\log_b x\end{aligned}$$

Notice that we used the product rule for logarithms to find a solution for the example above. By doing so, we have derived the **power rule for logarithms**, which says that the log of a power is equal to the exponent times the log of the base. Keep in mind that, although the input to a logarithm may not be written as a power, we may be able to change it to a power. For example,

$$100 = 10^2 \quad \sqrt{3} = 3^{\frac{1}{2}} \quad \frac{1}{e} = e^{-1}$$

### A General Note

#### The Power Rule for Logarithms

The power rule for logarithms can be used to simplify the logarithm of a power by rewriting it as the product of the exponent times the logarithm of the base.

$$\log_b(M^n) = n\log_b M$$

How To

**Given the logarithm of a power, use the power rule of logarithms to write an equivalent product of a factor and a logarithm.**

1. Express the argument as a power, if needed.
2. Write the equivalent expression by multiplying the exponent times the logarithm of the base.

Expanding a Logarithm with Powers

Expand  $\log_2 x^5$ .

The argument is already written as a power, so we identify the exponent, 5, and the base,  $x$ , and rewrite the equivalent expression by multiplying the exponent times the logarithm of the base.

$$\log_2(x^5) = 5\log_2 x$$

Try It

Expand  $\ln x^2$ .

## Logarithmic Properties

$2\ln x$

Rewriting an Expression as a Power before Using the Power Rule

Expand  $\log_3(25)$  using the power rule for logs.

Expressing the argument as a power, we get  $\log_3(25) = \log_3(5^2)$ .

Next we identify the exponent, 2, and the base, 5, and rewrite the equivalent expression by multiplying the exponent times the logarithm of the base.

$$\log_3(5^2) = 2\log_3(5)$$

Try It

Expand  $\ln\left(\frac{1}{x^2}\right)$ .

$$- 2\ln(x)$$

Using the Power Rule in Reverse

Rewrite  $4\ln(x)$  using the power rule for logs to a single logarithm with a leading coefficient of 1.

Because the logarithm of a power is the product of the exponent times the logarithm of the base, it follows that the product of a number and a logarithm can be written as a power. For the expression  $4\ln(x)$ , we identify the factor, 4, as the exponent and the argument,  $x$ , as the base, and rewrite the product as a logarithm of a power:  $4\ln(x) = \ln(x^4)$ .

Try It

Rewrite  $2\log_3 4$  using the power rule for logs to a single logarithm with a leading coefficient of 1.

$$\log_3 16$$

## Expanding Logarithmic Expressions

Taken together, the product rule, quotient rule, and power rule are often called “laws of logs.” Sometimes we apply more than one rule in order to simplify an expression. For example:



## Logarithmic Properties

$$\begin{aligned}\log_b\left(\frac{6x}{y}\right) &= \log_b(6x) - \log_by \\ &= \log_b6 + \log_bx - \log_by\end{aligned}$$

We can use the power rule to expand logarithmic expressions involving negative and fractional exponents. Here is an alternate proof of the quotient rule for logarithms using the fact that a reciprocal is a negative power:

$$\begin{aligned}\log_b\left(\frac{A}{C}\right) &= \log_b(AC^{-1}) \\ &= \log_b(A) + \log_b(C^{-1}) \\ &= \log_bA + (-1)\log_bC \\ &= \log_bA - \log_bC\end{aligned}$$

We can also apply the product rule to express a sum or difference of logarithms as the logarithm of a product.

With practice, we can look at a logarithmic expression and expand it mentally, writing the final answer. Remember, however, that we can only do this with products, quotients, powers, and roots—never with addition or subtraction inside the argument of the logarithm.

### Expanding Logarithms Using Product, Quotient, and Power Rules

Rewrite  $\ln\left(\frac{x^4y}{7}\right)$  as a sum or difference of logs.

First, because we have a quotient of two expressions, we can use the quotient rule:

$$\ln\left(\frac{x^4y}{7}\right) = \ln(x^4y) - \ln(7)$$

Then seeing the product in the first term, we use the product rule:

$$\ln(x^4y) - \ln(7) = \ln(x^4) + \ln(y) - \ln(7)$$

Finally, we use the power rule on the first term:

$$\ln(x^4) + \ln(y) - \ln(7) = 4\ln(x) + \ln(y) - \ln(7)$$

## Logarithmic Properties

Try It

Expand  $\log\left(\frac{x^2y^3}{z^4}\right)$ .

$$2\log x + 3\log y - 4\log z$$

Using the Power Rule for Logarithms to Simplify the Logarithm of a Radical Expression

Expand  $\log(\sqrt{x})$ .

$$\begin{aligned}\log(\sqrt{x}) &= \log x^{\left(\frac{1}{2}\right)} \\ &= \frac{1}{2}\log x\end{aligned}$$

Try It

Expand  $\ln(\sqrt[3]{x^2})$ .

$$\frac{2}{3}\ln x$$

Q&A

**Can we expand  $\ln(x^2 + y^2)$ ?**

*No. There is no way to expand the logarithm of a sum or difference inside the argument of the logarithm.*

Expanding Complex Logarithmic Expressions

Expand  $\log_6\left(\frac{64x^3(4x+1)}{(2x-1)}\right)$ .

We can expand by applying the Product and Quotient Rules.

$$\begin{aligned}\log_6\left(\frac{64x^3(4x+1)}{(2x-1)}\right) &= \log_6 64 + \log_6 x^3 + \log_6(4x+1) - \log_6(2x-1) && \text{Apply the Quotient Rule.} \\ &= \log_6 2^6 + \log_6 x^3 + \log_6(4x+1) - \log_6(2x-1) && \text{Simplify by writing 64 as } 2^6. \\ &= 6\log_6 2 + 3\log_6 x + \log_6(4x+1) - \log_6(2x-1) && \text{Apply the Power Rule.}\end{aligned}$$

Try It

Expand  $\ln\left(\frac{\sqrt{(x-1)(2x+1)^2}}{(x^2-9)}\right)$ .

$$\frac{1}{2}\ln(x-1) + \ln(2x+1) - \ln(x+3) - \ln(x-3)$$

## Condensing Logarithmic Expressions

We can use the rules of logarithms we just learned to condense sums, differences, and products with the same base as a single logarithm. It is important to remember that the logarithms must have the same base to be combined. We will learn later how to change the base of any logarithm before condensing.

How To

**Given a sum, difference, or product of logarithms with the same base, write an equivalent expression as a single logarithm.**

1. Apply the power property first. Identify terms that are products of factors and a logarithm, and rewrite each as the logarithm of a power.
2. Next apply the product property. Rewrite sums of logarithms as the logarithm of a product.
3. Apply the quotient property last. Rewrite differences of logarithms as the logarithm of a quotient.

Using the Product and Quotient Rules to Combine Logarithms

Write  $\log_3(5) + \log_3(8) - \log_3(2)$  as a single logarithm.

Using the product and quotient rules

$$\log_3(5) + \log_3(8) = \log_3(5 \cdot 8) = \log_3(40)$$

This reduces our original expression to

$$\log_3(40) - \log_3(2)$$

Then, using the quotient rule

$$\log_3(40) - \log_3(2) = \log_3\left(\frac{40}{2}\right) = \log_3(20)$$

Try It

## Logarithmic Properties

Condense  $\log 3 - \log 4 + \log 5 - \log 6$ .

$\log\left(\frac{3 \cdot 5}{4 \cdot 6}\right)$ ; can also be written  $\log\left(\frac{5}{8}\right)$  by reducing the fraction to lowest terms.

## Condensing Complex Logarithmic Expressions

Condense  $\log_2(x^2) + \frac{1}{2}\log_2(x - 1) - 3\log_2((x + 3)^2)$ .

We apply the power rule first:

$$\log_2(x^2) + \frac{1}{2}\log_2(x - 1) - 3\log_2((x + 3)^2) = \log_2(x^2) + \log_2(\sqrt{x - 1}) - \log_2((x + 3)^6)$$

Next we apply the product rule to the sum:

$$\log_2(x^2) + \log_2(\sqrt{x - 1}) - \log_2((x + 3)^6) = \log_2(x^2\sqrt{x - 1}) - \log_2((x + 3)^6)$$

Finally, we apply the quotient rule to the difference:

$$\log_2(x^2\sqrt{x - 1}) - \log_2((x + 3)^6) = \log_2\frac{x^2\sqrt{x - 1}}{(x + 3)^6}$$

## Rewriting as a Single Logarithm

Rewrite  $2\log x - 4\log(x + 5) + \frac{1}{x}\log(3x + 5)$  as a single logarithm.

We apply the power rule first:

$$2\log x - 4\log(x + 5) + \frac{1}{x}\log(3x + 5) = \log(x^2) - \log((x + 5)^4) + \log((3x + 5)^{x^{-1}})$$

Next we apply the product rule to the sum:

$$\log(x^2) - \log((x + 5)^4) + \log((3x + 5)^{x^{-1}}) = \log(x^2) - \log((x + 5)^4(3x + 5)^{x^{-1}})$$

Finally, we apply the quotient rule to the difference:

$$\log(x^2) - \log((x + 5)^4(3x + 5)^{x^{-1}}) = \log\left(\frac{x^2}{(x + 5)^4((3x + 5)^{x^{-1}})}\right)$$

## Logarithmic Properties

Try It

Rewrite  $\log(5) + 0.5\log(x) - \log(7x - 1) + 3\log(x - 1)$  as a single logarithm.

$$\log\left(\frac{5(x-1)^3\sqrt{x}}{(7x-1)}\right)$$

Try It

Condense  $4(3\log(x) + \log(x + 5) - \log(2x + 3))$ .

$$\log\frac{x^{12}(x+5)^4}{(2x+3)^4}; \text{ this answer could also be written } \log\left(\frac{x^3(x+5)}{(2x+3)}\right)^4.$$

## Applying of the Laws of Logs

Recall that, in chemistry,  $\text{pH} = -\log[H^+]$ . If the concentration of hydrogen ions in a liquid is doubled, what is the effect on pH?

Suppose  $C$  is the original concentration of hydrogen ions, and  $P$  is the original pH of the liquid. Then  $P = -\log(C)$ . If the concentration is doubled, the new concentration is  $2C$ . Then the pH of the new liquid is

$$\text{pH} = -\log(2C)$$

Using the product rule of logs

$$\text{pH} = -\log(2C) = -(\log(2) + \log(C)) = -\log(2) - \log(C)$$

Since  $P = -\log(C)$ , the new pH is

$$\text{pH} = P - \log(2) \approx P - 0.301$$

When the concentration of hydrogen ions is doubled, the pH decreases by about 0.301.

Try It

How does the pH change when the concentration of positive hydrogen ions is decreased by half?

The pH increases by about 0.301.

## Using the Change-of-Base Formula for Logarithms

Most calculators can evaluate only common and natural logs. In order to evaluate logarithms with a base other than 10 or  $e$ , we use the change-of-base formula to rewrite the logarithm as the quotient of logarithms of any other base; when using a calculator, we would change them to common or natural logs.

To derive the change-of-base formula, we use the one-to-one property and power rule for logarithms.

Given any positive real numbers  $M$ ,  $b$ , and  $n$ , where  $n \neq 1$  and  $b \neq 1$ , we show

$$\log_b M = \frac{\log_n M}{\log_n b}$$

Let  $y = \log_b M$ . By taking the log base  $n$  of both sides of the equation, we arrive at an exponential form, namely  $b^y = M$ . It follows that

$$\log_n(b^y) = \log_n M \quad \text{Apply the one-to-one property.}$$

$$y \log_n b = \log_n M \quad \text{Apply the power rule for logarithms.}$$

$$y = \frac{\log_n M}{\log_n b} \quad \text{Isolate } y.$$

$$\log_b M = \frac{\log_n M}{\log_n b} \quad \text{Substitute for } y.$$

For example, to evaluate  $\log_5 36$  using a calculator, we must first rewrite the expression as a quotient of common or natural logs. We will use the common log.

$$\log_5 36 = \frac{\log(36)}{\log(5)} \quad \text{Apply the change of base formula using base 10.}$$

$$\approx 2.2266 \quad \text{Use a calculator to evaluate to 4 decimal places.}$$

### A General Note

#### The Change-of-Base Formula

The change-of-base formula can be used to evaluate a logarithm with any base.

For any positive real numbers  $M$ ,  $b$ , and  $n$ , where  $n \neq 1$  and  $b \neq 1$ ,

$$\log_b M = \frac{\log_n M}{\log_n b}.$$

## Logarithmic Properties

It follows that the change-of-base formula can be used to rewrite a logarithm with any base as the quotient of common or natural logs.

$$\log_b M = \frac{\ln M}{\ln b}$$

and

$$\log_b M = \frac{\log M}{\log b}$$

How To

**Given a logarithm with the form  $\log_b M$ , use the change-of-base formula to rewrite it as a quotient of logs with any positive base  $n$ , where  $n \neq 1$ .**

1. Determine the new base  $n$ , remembering that the common log,  $\log(x)$ , has base 10, and the natural log,  $\ln(x)$ , has base  $e$ .
2. Rewrite the log as a quotient using the change-of-base formula
  - The numerator of the quotient will be a logarithm with base  $n$  and argument  $M$ .
  - The denominator of the quotient will be a logarithm with base  $n$  and argument  $b$ .

Changing Logarithmic Expressions to Expressions Involving Only Natural Logs

Change  $\log_5 3$  to a quotient of natural logarithms.

Because we will be expressing  $\log_5 3$  as a quotient of natural logarithms, the new base,  $n = e$ .

We rewrite the log as a quotient using the change-of-base formula. The numerator of the quotient will be the natural log with argument 3. The denominator of the quotient will be the natural log with argument 5.

$$\log_b M = \frac{\ln M}{\ln b}$$

$$\log_5 3 = \frac{\ln 3}{\ln 5}$$

Try It

Change  $\log_{0.5} 8$  to a quotient of natural logarithms.

$$\frac{\ln 8}{\ln 0.5}$$

Q&A

**Can we change common logarithms to natural logarithms?**

Yes. Remember that  $\log 9$  means  $\log_{10} 9$ . So,  $\log 9 = \frac{\ln 9}{\ln 10}$ .

Using the Change-of-Base Formula with a Calculator

Evaluate  $\log_2(10)$  using the change-of-base formula with a calculator.

According to the change-of-base formula, we can rewrite the log base 2 as a logarithm of any other base. Since our calculators can evaluate the natural log, we might choose to use the natural logarithm, which is the log base  $e$ .

$$\log_2 10 = \frac{\ln 10}{\ln 2} \quad \text{Apply the change of base formula using base } e.$$

$$\approx 3.3219 \quad \text{Use a calculator to evaluate to 4 decimal places.}$$

Try It

Evaluate  $\log_5(100)$  using the change-of-base formula.

$$\frac{\ln 100}{\ln 5} \approx \frac{4.6051}{1.6094} = 2.861$$

Media

Access these online resources for additional instruction and practice with laws of logarithms.

- [The Properties of Logarithms](#)
- [Expand Logarithmic Expressions](#)
- [Evaluate a Natural Logarithmic Expression](#)

**Key Equations**

<b>The Product Rule for Logarithms</b>	$\log_b(MN) = \log_b(M) + \log_b(N)$
<b>The Quotient Rule for Logarithms</b>	$\log_b\left(\frac{M}{N}\right) = \log_b M - \log_b N$
<b>The Power Rule for Logarithms</b>	$\log_b(M^n) = n \log_b M$
<b>The Change-of-Base Formula</b>	$\log_b M = \frac{\log_n M}{\log_n b} \quad n > 0, n \neq 1, b \neq 1$



## Key Concepts

- We can use the product rule of logarithms to rewrite the log of a product as a sum of logarithms. See [\[link\]](#).
- We can use the quotient rule of logarithms to rewrite the log of a quotient as a difference of logarithms. See [\[link\]](#).
- We can use the power rule for logarithms to rewrite the log of a power as the product of the exponent and the log of its base. See [\[link\]](#), [\[link\]](#), and [\[link\]](#).
- We can use the product rule, the quotient rule, and the power rule together to combine or expand a logarithm with a complex input. See [\[link\]](#), [\[link\]](#), and [\[link\]](#).
- The rules of logarithms can also be used to condense sums, differences, and products with the same base as a single logarithm. See [\[link\]](#), [\[link\]](#), [\[link\]](#), and [\[link\]](#).
- We can convert a logarithm with any base to a quotient of logarithms with any other base using the change-of-base formula. See [\[link\]](#).
- The change-of-base formula is often used to rewrite a logarithm with a base other than 10 and  $e$  as the quotient of natural or common logs. That way a calculator can be used to evaluate. See [\[link\]](#).

## Section Exercises

### Verbal

How does the power rule for logarithms help when solving logarithms with the form  $\log_b(\sqrt[n]{x})$ ?

Any root expression can be rewritten as an expression with a rational exponent so that the power rule can be applied, making the logarithm easier to calculate. Thus,

$$\log_b\left(x^{\frac{1}{n}}\right) = \frac{1}{n}\log_b(x).$$

What does the change-of-base formula do? Why is it useful when using a calculator?

### Algebraic

For the following exercises, expand each logarithm as much as possible. Rewrite each expression as a sum, difference, or product of logs.

$$\log_b(7x \cdot 2y)$$

$$\log_b(2) + \log_b(7) + \log_b(x) + \log_b(y)$$

## Logarithmic Properties

$$\ln(3ab \cdot 5c)$$

$$\log_b\left(\frac{13}{17}\right)$$

$$\log_b(13) - \log_b(17)$$

$$\log_4\left(\frac{x}{z/w}\right)$$

$$\ln\left(\frac{1}{4^k}\right)$$

$$-k\ln(4)$$

$$\log_2(y^x)$$

For the following exercises, condense to a single logarithm if possible.

$$\ln(7) + \ln(x) + \ln(y)$$

$$\ln(7xy)$$

$$\log_3(2) + \log_3(a) + \log_3(11) + \log_3(b)$$

$$\log_b(28) - \log_b(7)$$

$$\log_b(4)$$

$$\ln(a) - \ln(d) - \ln(c)$$

$$- \log_b\left(\frac{1}{7}\right)$$

$$\log_b(7)$$

$$\frac{1}{3}\ln(8)$$

For the following exercises, use the properties of logarithms to expand each logarithm as much as possible. Rewrite each expression as a sum, difference, or product of logs.

$$\log\left(\frac{x^{15}y^{13}}{z^{19}}\right)$$

## Logarithmic Properties

$$15\log(x) + 13\log(y) - 19\log(z)$$

$$\ln\left(\frac{a^{-2}}{b^{-4}c^5}\right)$$

$$\log(\sqrt{x^3y^{-4}})$$

$$\frac{3}{2}\log(x) - 2\log(y)$$

$$\ln\left(y\sqrt{\frac{y}{1-y}}\right)$$

$$\log(x^2y^3\sqrt[3]{x^2y^5})$$

$$\frac{8}{3}\log(x) + \frac{14}{3}\log(y)$$

For the following exercises, condense each expression to a single logarithm using the properties of logarithms.

$$\log(2x^4) + \log(3x^5)$$

$$\ln(6x^9) - \ln(3x^2)$$

$$\ln(2x^7)$$

$$2\log(x) + 3\log(x + 1)$$

$$\log(x) - \frac{1}{2}\log(y) + 3\log(z)$$

$$\log\left(\frac{xz^3}{\sqrt{y}}\right)$$

$$4\log_7(c) + \frac{\log_7(a)}{3} + \frac{\log_7(b)}{3}$$

For the following exercises, rewrite each expression as an equivalent ratio of logs using the indicated base.

$$\log_7(15) \text{ to base } e$$

$$\log_7(15) = \frac{\ln(15)}{\ln(7)}$$

## Logarithmic Properties

$\log_{14}(55.875)$  to base 10

For the following exercises, suppose  $\log_5(6) = a$  and  $\log_5(11) = b$ . Use the change-of-base formula along with properties of logarithms to rewrite each expression in terms of  $a$  and  $b$ . Show the steps for solving.

$\log_{11}(5)$

$$\log_{11}(5) = \frac{\log_5(5)}{\log_5(11)} = \frac{1}{b}$$

$\log_6(55)$

$\log_{11}\left(\frac{6}{11}\right)$

$$\log_{11}\left(\frac{6}{11}\right) = \frac{\log_5\left(\frac{6}{11}\right)}{\log_5(11)} = \frac{\log_5(6) - \log_5(11)}{\log_5(11)} = \frac{a - b}{b} = \frac{a}{b} - 1$$

### Numeric

For the following exercises, use properties of logarithms to evaluate without using a calculator.

$\log_3\left(\frac{1}{9}\right) - 3\log_3(3)$

$6\log_8(2) + \frac{\log_8(64)}{3\log_8(4)}$

3

$2\log_9(3) - 4\log_9(3) + \log_9\left(\frac{1}{729}\right)$

For the following exercises, use the change-of-base formula to evaluate each expression as a quotient of natural logs. Use a calculator to approximate each to five decimal places.

$\log_3(22)$

2.81359

$\log_8(65)$

$\log_6(5.38)$

## Logarithmic Properties

0.93913

$$\log_4\left(\frac{15}{2}\right)$$

$$\log_{\frac{1}{2}}(4.7)$$

- 2.23266

### Extensions

Use the product rule for logarithms to find all  $x$  values such that  $\log_{12}(2x + 6) + \log_{12}(x + 2) = 2$ . Show the steps for solving.

Use the quotient rule for logarithms to find all  $x$  values such that  $\log_6(x + 2) - \log_6(x - 3) = 1$ . Show the steps for solving.

$$x = 4; \text{ By the quotient rule: } \log_6(x + 2) - \log_6(x - 3) = \log_6\left(\frac{x+2}{x-3}\right) = 1.$$

Rewriting as an exponential equation and solving for  $x$  :

$$6^1 = \frac{x+2}{x-3}$$

$$0 = \frac{x+2}{x-3} - 6$$

$$0 = \frac{x+2}{x-3} - \frac{6(x-3)}{(x-3)}$$

$$0 = \frac{x+2-6x+18}{x-3}$$

$$0 = \frac{x-4}{x-3}$$

$$x = 4$$

Checking, we find that  $\log_6(4 + 2) - \log_6(4 - 3) = \log_6(6) - \log_6(1)$  is defined, so  $x = 4$ .

Can the power property of logarithms be derived from the power property of exponents using the equation  $b^x = m$ ? If not, explain why. If so, show the derivation.

Prove that  $\log_b(n) = \frac{1}{\log_n(b)}$  for any positive integers  $b > 1$  and  $n > 1$ .

## Logarithmic Properties

Let  $b$  and  $n$  be positive integers greater than 1. Then, by the change-of-base formula,

$$\log_b(n) = \frac{\log_n(n)}{\log_n(b)} = \frac{1}{\log_n(b)}.$$

Does  $\log_{81}(2401) = \log_3(7)$ ? Verify the claim algebraically.